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Non-Noetherian groups and primitivity of their group rings

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A ring R is (right) primitive provided it has a faithful irreducible (right) R-module. If non-trivial group G is finite or abelian, then the group ring KG over a field K can never be primitive. In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

(∗) For each finite subset M of non-identity elements of G, there exists a subset X of three elements of G such that $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$ implies $x_i = x_{i+1}$ for some i, where $g_i \in M$ and $x_i \in X$.

We can see that if G is countably infinite group and satisfies (∗), then KG is primitive for any field K. More generally, if G has a free subgroup whose cardinality is the same as that of G and satisfies (∗), then KG is primitive for any field K. As an application of this theorem, we improve or generalize [1]; we state the primitivity of group algebras of locally amalgamated free products.

1 Primitive group rings

Let R be a ring with the identity element (R need not be commutative). A ring R is right primitive if and only if there exists a faithful irreducible right R-module M_R, where M_R is irreducible provided it has no non-trivial submodules, and M_R is faithful provided the annihilator of it is zero. The above definition is equivalent to the following: There exists a maximal right ideal ρ in R which contains no non-trivial ideals.

Let KG be the group ring of a group G over a field K. If non-trivial group G is finite or abelian, then the group ring KG over a field K can never be primitive. The first example of primitive group rings was offered by Formanek and Snider [5] in 1972. After that, many examples of primitive group rings were constructed. In 1978, Domanov [2], Farkas-Passman [3] and Roseblade [10] gave the complete solution for primitivity of group rings of polycyclic-by-finite groups.

Theorem 1.1. *(Domanov[2], Farkas-Passman[3],Roseblade[10]) Let G be a non-trivial polycyclic-by-finite group. Then KG is primitive if and only if $\Delta(G) = 1$*

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and K is non-absolute, where $\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ and K is absolute if it is algebraic over a finite field.

Polycyclic-by-finite groups are belong to the class of noetherian groups. Almost all other infinite groups are belong to the class of non-noetherian groups, because it is not easy to find a noetherian group which is not polycyclic-by-finite [8]. As is well known, if KG is noetherian then G is also noetherian, but the converse is not true generally. A group of the class of non-noetherian groups which is, in particular, finitely generated has often non-abelian free subgroups; for instance, a free group, a locally free group, a free product, an amalgamated free product, an HNN-extension, a Fuchsian group, a one relator group, etc (a free Burnside group is not the case, though). Primitivity of group rings of some of those groups have been obtained gradually: In 1973, primitivity of group rings of free products [4]. In 1989, primitivity of group rings of amalgamated free products [1]. In 2007, primitivity of group rings of ascending HNN-extensions of free groups [6]. In 2011, primitivity of group rings of locally free groups [7]. However, much of them remains unknown. In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

($*$) For each finite subset M of non-identity elements of G, there exists a subset X of three elements of G such that $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$ implies $x_i = x_{i+1}$ for some i, where $g_i \in M$ and $x_i \in X$.

We can see that if G is countably infinite group and satisfies ($*$), then KG is primitive for any field K. More generally, we can get the following theorem:

Theorem 1.2. *Let G be a non-trivial group which has a free subgroup whose cardinality is the same as that of G. Suppose that G satisfies the condition ($*$). If R is a domain with $|R| \leq |G|$, then the group ring RG of G over R is primitive. In particular, the group algebra KG is primitive for any field K.*

As an application of the theorem, we generalize [1]; we state the primitivity of group algebras of locally amalgamated free products.

One of the main method to prove Theorem 1.2 is a graph theoretic method which is called SR-graph theory.

2 Theory of SR-graphs

Let $\mathcal{G} = (V, E)$ denote a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. A finite sequence $v_0e_1v_1\cdots e_pv_p$ whose terms are alternately elements e_q's in E and

v_q's in V is called a path of length p in $\mathcal{G}$ if $v_q \neq v_{q'}$ for any $q, q' \in \{0, 1, \cdots, p\}$ with $q \neq q'$; it is often simply denoted by $v_0 v_1 \cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's $(1 \leq i \leq m)$ for some $m > 0$ such that $v, w \in V$ are connected if and only if both v and w belong to the same set C_i. The subgraph (C_i, E_i) of $\mathcal{G}$ generated by C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components. For $v \in V$, we denote by $C(v)$ the component of $\mathcal{G}$ which contains the vertex v.

Definition 2.1. *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{H}$ and v itself: $U(v) = \{w \in V \mid vw \in F\} \cup \{v\}$. A triple (V, E, F) is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions:*

(SR1) *For any $v \in V$, $C(v) \cap U(v) = \{v\}$.*
(SR2) *Every component of $\mathcal{G}$ is a complete graph.*

If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph (V, E, F) is called a proper SR-graph.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $\mathcal{S} = (V, E, F)$ be an SR-graph. We say $\mathcal{S}$ is connected if the graph $(V, E \cup F)$ is connected.

Definition 2.2. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1 w_1 v_2 w_2, \cdots, v_p w_p v_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length p in $\mathcal{S}$ if either $e_q = v_q w_q \in E$ and $f_q = w_q v_{q+1} \in F$ or $f_q = v_q w_q \in F$ and $e_q = w_q v_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively. If, in addition, it is a cycle in $(V, E \cup F)$; namely, $v_{p+1} = v_1$, then it is an SR-cycle of length p in $\mathcal{S}$.*

To prove Theorem 1.2, we use some results for SR-graphs and apply them to the Formanek's method. We can give Formanek's method, as follows:

Proposition 2.3. (See [4]) *Let RG be the group ring of a group G over a domain R with identity. Suppose that the cardinality of R is not larger than that of G. If for each non-zero $a \in RG$, there exists an element $\varepsilon(a)$ in the ideal $RGaRG$ generated by a such that the right ideal $\rho = \sum_{a \in RG \setminus \{0\}} (\varepsilon(a) + 1) RG$ is proper; namely, $\rho \neq RG$, then RG is primitive.*

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper. Now, ρ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since ρ is generated by the elements of form $(\varepsilon(a)+1)$ with $a \neq 0$, r has the presentation, $r = \sum_{(a,b)\in\Pi}(\varepsilon(a)+1)b$, where Π is a subset which consists of finite number of elements of $RG \times RG$ both of whose components are non-zero. Moreover, $\varepsilon(a)$ and b are linear combinations of elements of G, and so we have

$$r = \sum_{(a,b)\in\Pi} \sum_{g\in S_a, h\in T_b} (\alpha_g \beta_h gh + \beta_h h), \tag{1}$$

where S_a and T_b are the support of $\varepsilon(a)$ and b respectively and both α_g and β_h are elements in K. In the above presentation (1), if there exists gh such that $gh \neq 1$ and does not coincide with the other $g'h'$'s and h''s, then $r \neq 1$ holds. (Strictly speaking: Let $\Omega_{ab} = S_a \times T_b$. If there exist $(a,b) \in \Pi$ and (g,h) in Ω_{ab} with $gh \neq 1$ such that $gh \neq g'h'$ and $gh \neq h'$ for any $(c,d) \in \Pi$ and for any (g',h') in Ω_{cd} with $(g',h') \neq (g,h)$, then $r \neq 1$ holds.)

On the contrary, if $r = 1$, then for each gh in (1) with $gh \neq 1$, there exists another $g'h'$ or h' in (1) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist $g_{2i-1}h_i$ and $g_{2i}h_{i+1}$ $(i = 1, \cdots, m)$ in (1) such that the following equations hold:

$$\begin{aligned} g_1h_1 = {} & g_2h_2, \\ & g_3h_2 = g_4h_3, \\ & \qquad \ddots \\ & \qquad g_{2m-1}h_m = g_{2m}h_{m+1} \quad \text{and} \quad h_{m+1} = h_1. \end{aligned} \tag{2}$$

Eliminating h_i's in the above, we can see that these equations imply the equation $g_1g_2^{-1}\cdots g_{2m-1}g_{2m}^{-1} = 1$. If we can choose $\varepsilon(a)$'s so that their supports g_i's never satisfy such an equation, then we can prove that $r \neq 1$ holds by contradiction. We need therefore only to see when supports g's of $\varepsilon(a)$'s satisfy equations as described in (2).

By making use of graph theoretic considerations, we can state the following theorems:

Theorem 2.4. *Let $\mathcal{S} = (V,E,F)$ be an SR-graph and let ω_E and ω_F be, respectively, the number of components of $\mathcal{G} = (V,E)$ and $\mathcal{H} = (V,F)$. Suppose that every component of $\mathcal{H} = (V,F)$ is a complete graph and $\mathcal{S}$ is connected. Then $\mathcal{S}$ has an SR-cycle if and only if $\omega_E + \omega_F < |V| + 1$.*

In particular, if $\mathcal{S}$ is proper and $\alpha \leq \gamma$ then $\mathcal{S}$ has an SR-cycle.

We next consider the case that every component $\mathcal{H}_i = (V_i, F_i)$ of $\mathcal{H}$ is a complete k-partite graph $K_{m_1,\cdots,m_k}$. Let $\mu(\mathcal{H}_i)$ be the maximum number in $\{m_1, \cdots, m_k\}$. For $W \subseteq V$, $I_{\mathcal{G}}(W)$ denotes the set of isolated vertices in W on

$\mathcal{G}$; namely $I_{\mathcal{G}}(W) = \{v \in W \mid d_{\mathcal{G}}(v) = 0\}$. $\mathfrak{C}(V)$ denotes the set of components of V on $\mathcal{H} = (V, F)$.

Theorem 2.5. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $\mathfrak{C}(V) = \{V_1, \cdots, V_n\}$ with $n > 0$. Suppose that every component $\mathcal{H}_i = (V_i, F_i)$ of $\mathcal{H}$ is a complete k-partite graph with $k > 1$, where k is depend on $\mathcal{H}_i$. If $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in \{1, \cdots, n\}$ and $|I_{\mathcal{G}}(V)| \leq n$ then $\mathcal{S}$ has an SR-cycle.*

3 Proof of the main theorem

Let G be a group and $M_1, \cdots, M_n$ non-empty subsets of G which do not include the identity element. We say $M_1, \cdots, M_n$ are mutually reduced in G if for each finite elements $g_1, \cdots, g_m$ in the union of M_i's, $g_1 \cdots g_m = 1$ implies both g_i and g_{i+1} are in the same M_j for some i and j. If $M_1 = \{x_1^{\pm 1}\}, \cdots, M_m = \{x_m^{\pm 1}\}$ in the above, then we say simply $x_1, \cdots, x_m$ are mutually reduced.

In this section, we shall prove Theorem 1.2 after preparing three lemmas.

Lemma 3.1. (See [9, Theorem 2]) *Let K' be a field and G a group. If $\Delta(G)$ is trivial and $K'G$ is primitive, then for any field extension K of K', KG is primitive.*

Lemma 3.2. *Let G be a non-trivial group, $m > 0$ and $n > 0$. For non-trivial distinct elements f_{ij}'s $(i = 1, 2, 3,\ j = 1, \cdots, m)$ in G and for distinct elements g_i's $(i = 1, \cdots, n)$ in G, we set*

$$\begin{array}{ll} S & = \bigcup_{i=1}^{3} S_i, \ \text{where } S_i = \{f_{ij} \mid 1 \leq j \leq m\}, \\ T & = \{g_i \mid 1 \leq i \leq n\}, \\ V & = S \times T, \\ M_i & = \{f_{ij}^{\pm 1},\ f_{ij}^{-1} f_{ik} \mid j, k = 1, 2, \cdots, m,\ j \neq k\}\ (i = 1, 2, 3), \\ I & = \{(f, g) \in V \mid fg \neq f'g' \ \text{for any } (f', g') \in V \text{ with } (f', g') \neq (f, g)\}. \end{array}$$

Then if M_1, M_2 and M_3 are mutually reduced, then $|I| > n$.

Lemma 3.3. *Let G be a non-trivial group and $n > 0$. For each $i = 1, 2, \cdots, n$, let $f_{i1}, \cdots, f_{im_i}$ be distinct $m_i > 0$ elements of G; $f_{ip} \neq f_{iq}$ for $p \neq q$, and let x_{ij} $(1 \leq i \leq n, 1 \leq j \leq 3)$ be distinct elements in G. we set*

$$\begin{array}{ll} S & = \bigcup_{i=1}^{3} S_i, \ \text{where } S_i = \{f_{ij} \mid 1 \leq j \leq m_i\}, \\ X & = \bigcup_{i=1}^{n} X_i, \ \text{where } X_i = \{x_{ij} \mid 1 \leq j \leq 3\}, \\ V & = \bigcup_{i=1}^{n} V_i, \ \text{where } V_i = X_i \times S_i, \\ I & = \{(x, f) \in V \mid xf \neq x'f' \ \text{for any } (x', f') \in V \text{ with } (x', f') \neq (x, f)\}. \end{array}$$

If x_{ij}'s are mutually reduced elements, then $|I| > m$, where $m = m_1 + \cdots + m_n$.

Proof of Theorem 1.2. Let B be the basis of a free subgroup of G whose cardinality is the same as that of G. Then we may assume that the cardinality of B is also same as G, that is, $|B| = |G|$. In addition, since $|R| \leq |G|$, we have that $|B| = |RG|$. We can divide B into three subsets B_1, B_2 and B_3 each of whose cardinality is $|B|$. It is then obvious that the elements in B are mutually reduced. Let φ be a bijection from B to $RG \setminus \{0\}$ and σ_s a bijection from B to B_s, $s = 1, 2, 3$.

For $b \in B$, let $\varphi(b) = \sum_{f \in F_b} \alpha_f f$, where $\alpha_f \in R$ and F_b is the support of $\varphi(b)$. We set

$$M_b = \{f^{\pm 1},\ f^{-1}f' \mid f, f' \in F_b, f \neq f'\}.$$

Since G satisfies the condition $(*)$, there exist $x_{b1}, x_{b2}, x_{b3} \in G$ such that $M_b^{x_{bt}} = \{x_{bt}^{-1} f^{\pm 1} x_{bt},\ x_{bt}^{-1} f^{-1} f' x_{bt} \mid f, f' \in F_b,\ f \neq f'\}$ $(t = 1, 2, 3)$ are mutually reduced. We here define $\varepsilon(b)$ and $\varepsilon^1(b)$ by

$$\varepsilon(b) = \sum_{s=1}^{3} \sum_{t=1}^{3} \sigma_s(b) x_{bt}^{-1} \varphi(b) x_{bt} \quad \text{and} \quad \varepsilon^1(b) = \varepsilon(b) + 1. \tag{3}$$

Note that $\varepsilon(b)$ is an element in the ideal of RG generated by $\varphi(b)$. Let $\rho = \sum_{b \in B} \varepsilon^1(b) RG$ be the right ideal generated by $\varepsilon^1(b)$ for all $b \in B$. If $w \in \rho$, then we can express w by

$$w = \sum_{b \in A} \varepsilon^1(b) u_b = \sum_{b \in A} (\varepsilon(b) u_b + u_b) \tag{4}$$

for some non-empty finite subsets A of B and u_b in RG. In view of Proposition 2.3, in order to prove that RG is primitive, we need only show that ρ is proper; $\rho \neq RG$. To do this, it suffices to show that $w \neq 1$.

Let $u_b = \sum_{h \in H_b} \beta_h h$, where H_b is the support of u_b. Substituting this and $\varphi(b) = \sum_{f \in F_b} \alpha_f f$ into (3), we obtain the following expression of $\varepsilon(b) u_b$:

$$\varepsilon(b) u_b = \sum_{s=1}^{3} \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_{bs} x_{bt}^{-1} f x_{bt} h, \text{ where } y_{bs} = \sigma_s(b). \tag{5}$$

In what follows, for the sake of convenience, we represent $y_{bs} x_{bt}^{-1} f x_{bt} h$ by $y_s x_t^{-1} f x_t h$, and we note that y_s and x_t are depend on $b \in B$. For $s = 1, 2, 3$, we here set

$$E_{bs} = \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_s \xi(x_t, f, h), \text{ where } \xi(x_t, f, h) = x_t^{-1} f x_t h. \tag{6}$$

That is, $\varepsilon(b) u_b = E_{b1} + E_{b2} + E_{b3}$. We can see that there exist more than $|H_b|$ isolated elements in the expression (6) of E_{bs} for each $s = 1, 2, 3$. Strictly speaking, if we set $X_b = \{x_1, x_2, x_3\}$, $\Gamma_b = X_b \times F_b \times H_b$ and

$$I_s = \{(x_t, f, h) \quad | \ (x_t, f, h) \in \Gamma_b,\ \xi(x_t, f, h) \neq \xi(x_p, f', h') \\ \text{for any } (x_p, f', h') \in \Gamma_b \text{ with } (x_p, f', h') \neq (x_t, f, h)\},$$

then $|I_s| > |H_b|$. In fact, since $M_b^{x_{bt}}$ $(t = 1, 2, 3)$ are mutually reduced, it follows from lemma 3.2 that $|I_s| > |H_b|$.

Now, we shall see that $w \neq 1$ holds, where w as in (4). In (4), we set that $w_1 = \sum_{b \in A} \varepsilon(b) u_b$ and $w_2 = \sum_{b \in A} u_b$. We have then that

$$w_1 = \sum_{b \in A} \sum_{s=1}^{3} E_{bs} \quad \text{and} \quad w = w_1 + w_2.$$

Let $Supp(E_{bs})$ be the support of E_{bs} and let $m_b = |Supp(E_{b1})|$. We should note that $|Supp(E_{bs})| = m_b$ for all $s = 1, 2, 3$. It is obvious that $m_b \geq |I_s|$, and so $m_b > |H_b|$ by the above. Since y_{bs} $(b \in A, 1 \leq s \leq 3)$ are mutually reduced, by virtue of Lemma 3.3, we have $|Supp(w_1)| > \sum_{b \in A} m_b$. Moreover we have that

$$\begin{aligned} |Supp(w)| &\geq |Supp(w_1)| - |Supp(w_2)| \\ &> \sum_{b \in A} m_b - \sum_{b \in A} |H_b| \\ &> 0, \end{aligned}$$

which implies $|Supp(w)| \geq 2$. In particular, $w \neq 1$. We have thus seen that RG is primitive.

Finally, we shall show that KG is primitive for any field K. Let K' be a prime field. Since G satisfies $(*)$ and $|K'| \leq |G|$, we have already seen that $K'G$ is primitive. In view of Lemma 3.1, we need only show that $\Delta(G) = 1$.

Let g be a non-identity element in G. We can see that there exist infinite conjugate elements of g. In fact, if it is not true, then the set M of conjugate elements of g in G is a finite set. Since G satisfies $(*)$, for M, there exists $x_1, x_2 \in G$ such that M^{x_1} and M^{x_2} are mutually reduced. Since g is in M, $(x_1^{-1} g x_1)(x_2^{-1} f x_2)^{-1} \neq 1$ for any $f \in M$, and thus $x_1^{-1} g x_1 \neq x_2^{-1} f x_2$. Hence $(x_1 x_2^{-1})^{-1} g (x_1 x_2^{-1}) \neq f$ for all $f \in M$, which implies a contradiction $x^{-1} g x \notin M$, where $x = x_1 x_2^{-1}$. This completes the proof of theorem. □

4 An application of the main theorem

In what follows in this section, let $A *_H B$ be the free product of A and B with H amalgamated, and suppose that $A \neq H \neq B$. For $x, u_1, \cdots, u_n \in A *_H B$, we write $x \equiv u_1 \cdots u_n$ or $x^\rho = u_1 \cdots u_n$ provided that $u_1 \cdots u_n$ is a reduced form for x, that is, $x = u_1 \cdots u_n$, $u_i \notin H$, $u_i \in A \cup B$, u_i and u_{i+1} are not both in A or both in B. For x as above, n is called the length of x and is denoted here by $l(x)$. If $x \in H$, we define $l(x) = 0$. For $x, U, V, W \in A *_H B$, we also write $x \equiv UVW$ provided that $x = UVW$ and $x \equiv u_1 \cdots u_n v_1 \cdots v_m w_1 \cdots w_l$ where $U \equiv u_1 \cdots u_n$,

$V \equiv v_1 \cdots v_m$ and $W \equiv w_1 \cdots w_l$. For a set M of finite elements of G and an element $x \in G$, we denote $\{x^{-1}fx \mid f \in M\}$ by M^x.

We consider the following condition on $A *_H B$:

(†) $B \neq H$ and there exist elements a and a_* in $A \setminus H$ such that $aa_* \neq 1$ and $a^{-1}Ha \cap H = 1$.

In this section, as an application of the main theorem, we generalize [1] and state the primitivity of group algebras of locally amalgamated free products:

Theorem 4.1. *Let R be a domain (i.e. a ring with no zero divisors) and G a non-trivial group which has a free subgroup whose cardinality is the same as that of G. Suppose that for each finite elements $f_1, \cdots, f_n$ in G, there exists a subgroup N containing $f_1, \cdots, f_n$ such that N is isomorphic to $A *_H B$ which satisfies the condition (†).*

Then the group ring RG is primitive provided $|R| \leq |G|$. In particular, KG is primitive for any field K.

If $A \neq H \neq B$, then $A *_H B$ has always a countable free subgroup. Hence, in the above theorem, the assumption on existence of a free subgroup is needed only in the case of $|G| > \aleph_0$.

In view of Theorem 1.2, to prove the theorem above, we need only show that G satisfies the condition $(*)$ described in the previous section. In the above theorem, it is supposed that for each finite elements $f_1, \cdots . f_n$ in G, there exists a subgroup $N = A *_H B$ containing $f_1, \cdots . f_n$ such that N satisfies (†). Hence it suffices to show that $A *_H B$ has always the property $(*)$ provided it satisfies (†). In fact, if $b \in B \setminus H$ and $a, a_* \in A$ which satisfy $aa_* \neq 1$ and $a^{-1}Ha \cap H = 1$, then for $i = 1, 2, 3$,

$$x_i = (b^{-1}a)^{\omega_i} a_* b^{-1} a_*^{-1} (b^{-1}a)^{\omega_i} \quad \text{if } aa_* \notin H \tag{7}$$

$$x_i = (b^{-1}a^{-1})^{\omega_i} a_*^{-1} b^{-1} a_* (b^{-1}a^{-1})^{\omega_i} \quad \text{if } a_*a \notin H \tag{8}$$

are desired elements in $A *_H B$; namely, for $M = \{f_1, \cdots . f_n\}$, M^{x_i} $(i = 1, 2, 3)$ are mutually reduced, where $\omega_i = l + i$ for $i \in \{1, 2, 3\}$ and l is the maximum number in the set $\{l(f_i) \mid 1 \leq i \leq n\}$. We shall confirm this after preparing a lemma.

Lemma 4.2. *Let $G = A *_H B$. Suppose that G satisfies (†), and let a be an element as in (†) above. Let $1 \neq f \in G$ with $l(f) = l$ and $W = (a^{-1}b)^m f (b^{-1}a)^m$, where m is a positive integer and $b \in B \setminus H$.*

If $m > l + 1$, then a reduced form of W is of form

$$W \equiv (a^{-1}b)V(b^{-1}a) \text{ for some reduced form word } V, \tag{9}$$

otherwise $W \equiv (b^{-1}a)^{\pm k}$ for some $k > 0$.

Proof. Let f in G with $l(f) = l$. Then a reduced form f^ρ of f is one of following forms:

(T0)	$f^\rho = h$ if $l = 0$,
(T1)	$f^\rho = \alpha_1\beta_2\cdots\beta_{l-1}\alpha_l$,
(T2)	$f^\rho = \alpha_1\beta_2\cdots\alpha_{l-1}\beta_l$,
(T3)	$f^\rho = \beta_1\alpha_2\cdots\alpha_{l-1}\beta_l$,
(T4)	$f^\rho = \beta_1\alpha_2\cdots\beta_{l-1}\alpha_l$,

where $h \in H$, $\alpha_i \in A \setminus H$ and $\beta_i \in B \setminus H$.

In order to see that the assertions hold, it suffices to show when f^ρ is of the above forms; (T0)-(T4).

Let $W = (a^{-1}b)^m f^\rho (b^{-1}a)^m$. If f^ρ is of form (T1), then it is trivial that W^ρ is of form (9). We may therefore assume that f^ρ is not of form (T1).

We first suppose that f^ρ is of form (T2). It suffices to show that W_1^ρ is of form (9), otherwise $W_1 \equiv (a^{-1}b)^k$, where $k > 0$. We prove it by induction on l.

Let $l = 0$; thus $f^\rho = h \neq 1$ is of form (T0). We set $b' = bhb^{-1}$ and $a' = a^{-1}b'a$. Then $b' \neq 1$ because of $h \neq 1$. If $b' \notin H$, then $W \equiv (a^{-1}b)^{m-1}a^{-1}b'a(b^{-1}a)^{m-1}$ is of of form (9), and therefore we may assume that $b' \in H$. In this case, if $a' \in H$ then $a' = 1$ by (†), which implies a contradiction; $b' = 1$. Hence we have that $a' \notin H$ and thus $a' \in A \setminus H$, which implies that $W \equiv (a^{-1}b)^{m-1}a'(b^{-1}a)^{m-1}$ is of form (9).

Now let $l > 0$ and suppose that the assertion holds provided that the length of f^ρ is less than l. Since f^ρ is of form (T2), in this case, $l \geq 2$. If $\beta_l b^{-1} \notin H$, then the assertion is trivial, and so we may assume that $\beta_l b^{-1} \in H$ and also that $\alpha_{l-1}\beta_l b^{-1}a \in H$. Let $\alpha'_{l-1} = \alpha_{l-1}\beta_l b^{-1}a$. If $l = 2$ and $\alpha'_{l-1} = 1$, then $W = (a^{-1}b)^m(b^{-1}a)^{m-1}$, and hence $W \equiv (a^{-1}b)$. We may therefore assume that $\alpha'_{l-1} \neq 1$ for $l = 2$. We set $f' = \alpha'_{l-1}$ for $l = 2$ and $f' = \alpha_1\beta_2\cdots\beta'_{l-2}$ for $l > 2$, where $\beta'_{l-2} = \beta_{l-2}\alpha'_{l-1} \in B \setminus H$. Let $W' = (a^{-1}b)^{m-1}f'(b^{-1}a)^{m-1}$. In the case of $l = 2$, since $l(f') = 0$, we have already seen that a reduced form of W' is of form (9). In the case of $l > 2$, f' is of form (T2). Since $l(f') < l$ and $m - 1 > l(f) = l(f') + 2 > l(f') + 1$, it follows from our inductive hypothesis that a reduced form of W' is of form (9), otherwise $W' \equiv (a^{-1}b)^p$, where $p > 0$. Since $W = a^{-1}bW'$, if W^ρ is not of form (9), then $W \equiv (a^{-1}b)^{p+1}$. We have thus seen that the assertion of lemma holds when f^ρ is of form (T2).

If f^ρ is of form (T4), then $(f^\rho)^{-1}$ is of form (T2). Therefore, replacing W by W^{-1}, it follows from the above that the assertion of lemma holds when f^ρ is of form (T4). So the remaining case is that f^ρ is of form (T3).

Suppose that f^ρ is of form (T3). We shall show in this case that W^ρ is of form (9). It is proved by induction on l.

Let $l = 1$; thus $f^\rho = \beta_1$. Let $b' = b\beta_1 b^{-1}$ and $a' = a^{-1}b'a$. Then $b' \neq 1$ because of $\beta_1 \neq 1$. Similarly as above, we may assume that $b' \in H$. In this case, $a' \in A \backslash H$ by (†) and $W \equiv (a^{-1}b)^{m-1}a'(b^{-1}a)^{m-1}$ is of form (9) because of $m > 2$.

Now, let $l > 1$ and suppose that W^ρ is of form (9) provided that the length of f^ρ is less than l. Since f^ρ is of form (T3), in this case, $l > 2$. Let $\beta_1' = b\beta_1$ and $\alpha_2' = a^{-1}\beta_1'\alpha_2$. As we saw above, we may assume that $\beta_1' \in H$ and also $\alpha_2' \in H$. Let $\beta_3' = \alpha_2'\beta_3$, and then $\beta_3' \in B \setminus H$. We set that $f' = \beta_3'\alpha_4 \cdots \alpha_{l-1}\beta_l$ and $W' = (a^{-1}b)^{m-1}f'(b^{-1}a)^{m-1}$. Since $l(f') = l - 2 < l$ and $m - 1 > l(f) = l(f') + 2 > l(f') + 1$, it follows from our inductive hypothesis that a reduced form of W' is of form (9), and so is W because of $W = W'b^{-1}a$. This complete the proof of the lemma. □

Proof of Theorem 4.1. Let $M = \{f_1, \cdots, f_n\}$ be a set of finite non-trivial elements in G. By the assumption of the statement, there exists a subgroup N with $M \subset N$ such that $N \simeq A *_H B$ which satisfies (†). As was mentioned at the beginning of this section, it suffices to show that M^{x_i} $(i = 1, 2, 3)$ are mutually reduced, where x_i $(i = 1, 2, 3)$ are as in (7) and (8). Replacing a and a_* in (7) by a^{-1} and a_*^{-1} respectively, we can get the case of (8), and so we shall show only in the case of (7); namely, we let $x_i = (b^{-1}a)^{\omega_i}a_*b^{-1}a_*^{-1}(b^{-1}a)^{\omega_i}$ and suppose $aa_* \notin H$.

Let $g_{ip} = x_i^{-1}f_px_i$ $(p = 1, \cdots, n)$ are the elements in M^{x_i}. Since $\omega_i = l + i$ for $i \in \{1, 2, 3\}$ and l is the maximum number in the set $\{l(f_i) \mid 1 \leq i \leq n\}$, by virtue of Lemma 4.2, for each $i \in \{1, 2, 3\}$ and each $p \in \{1, 2, \cdots, n\}$, the reduced form W_{ip} of $(a^{-1}b)^{\omega_i}f_p(b^{-1}a)^{\omega_i}$ is either $(b^{-1}a)^{\pm k}$ for some $k > 0$ or $(a^{-1}b)V_{ip}(b^{-1}a)$ for some reduced form word V_{ip}. In either case, since $aa_* \in A \setminus H$, we may consider that $a_*^{-1}W_{ip}a_*$ is a reduced form word. We set $A_{ip} \equiv a_*^{-1}W_{ip}a_*$. We have then that

$$g_{ip} \equiv X_i^{-1}A_{ip}X_i, \tag{10}$$

where $X_i = b^{-1}a_*^{-1}(b^{-1}a)^{\omega_i}$. If $i \neq j$, say $i > j$, then a reduced form B_{ij} of $X_iX_j^{-1}$ is $b^{-1}a_*^{-1}(b^{-1}a)^{\omega_i - \omega_j}a_*b$. Therefore we have

$$g_{ip}g_{jq} \equiv X_i^{-1}A_{ip}B_{ij}A_{jq}X_j. \tag{11}$$

Now, let $g = g_1 \cdots g_k$ be any finite product of g_i's in $\bigcup_{j=1}^{3} M^{x_j}$. If both of g_i and g_{i+1} are not in the same M^{x_j}, since the reduced form of g_i is of form (10), by noting that g_ig_{i+1} has the reduced form of (11), it can be easily seen by induction on k that $g \equiv X_1^{-1}UX_k$ for some reduced form word U with $U \neq 1$ in G. Hence, in particular, $g \neq 1$. We have thus seen that M^{x_i}'s are mutually reduced. This completes the proof of the theorem. □

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